Journal of Approximation Theory **99**, 230–241 (1999)
Article ID jath.1998.3317, available online at http://www.idealibrary.com on IDEAL®

Balayage and Convergence of Rational Interpolants

Amiran Ambroladze

Department of Mathematics, Tbilisi University, Republic of Georgia, and
Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden
E-mail: Amiran.Ambroladze@mathdept.umu.se

and

Hans Wallin

Department of Mathematics, University of Umeå, S-901 87 Umeå, Sweden
E-mail: Hans.Wallin@mathdept.umu.se

Communicated by Vilmos Totik

Received January 15, 1998; accepted August 11, 1998

We investigate the following problem: For which open simply connected domains do there exist interpolation schemes (a set of interpolation points) such that for any analytic function defined in the domain the corresponding interpolating polynomials converge to the function when the degree of the polynomials tends to infinity? We also study similar problems for rational interpolants. These problems are connected to the balayage (sweeping out) problems of measures.

1. INTRODUCTION

Let D be a bounded simply connected domain in the complex plane $\mathbb{C}$ and let, for each positive integer n, $A_n = \{a_{nj}\}_{j=0}^{n} \subset D$ be a set of points, the interpolation points. Let f be an analytic function in D. Then there exists a unique polynomial P_n of degree at most n interpolating to f at the points of A_n, i.e., the points a_{nj}, $0 \leqslant j \leqslant n$, are zeros of $f - P_n$, counting multiplicities. We ask the following question: Is it possible to choose A_n, independently of f, so that P_n converges to f in D, as $n \to \infty$, for all analytic functions f in D? For instance, we shall see that the answer is yes if the boundary ∂D of D is an analytic curve (see Example 2, Section 4) but no if D is the interior of a rectangle (Example 3, Section 4).

The convergence problem stated above turns out to be connected to the balayage (sweeping out) problem of measures in the following way. For

0021-9045/99 $30.00

each $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ let δ_a denote the Dirac measure at a, i.e., the probability measure with mass 1 at a. We introduce the *normalized point counting measure* of A_n:

$$\alpha_n = \frac{1}{n+1} \sum_{j=0}^{n} \delta_{a_{nj}}. \tag{1}$$

Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ converges in the weak star sense to a measure α, i.e., that $\int \varphi \, d\alpha_n \to \int \varphi \, d\alpha$ for every continuous function φ on $\overline{\mathbb{C}}$, or that a subsequence of $\{\alpha_n\}$ converges to α. By weak star compactness (see for instance [St-To]) there always exists such a measure α. We shall see in Section 4 that the solution to the convergence problem for interpolating polynomials depends on the following question: Is it possible to choose the interpolation points A_n so that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D and the balayage α' of α onto ∂D coincides with the equilibrium measure τ of total mass 1 on ∂D for logarithmic potentials, $\alpha' = \tau$? We refer to Section 2 for the definition of balayage. However, $\tau = \delta'_\infty$, the balayage onto ∂D of the Dirac measure δ_∞ at infinity (see Subsection 2.1). Consequently, the condition $\alpha' = \tau$ can be written $\alpha' = \delta'_\infty$. Furthermore, the condition that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D, implies that $\operatorname{supp}(\alpha) \subset D$ where $\operatorname{supp}(\alpha)$ stands for the support of α, i.e., the smallest closed subset of $\overline{\mathbb{C}}$ outside which α is zero. Summing up, we are led to the following balayage problem: For a given D, does there exist a probability measure α, $\operatorname{supp}(\alpha) \subset D$, such that $\alpha' = \delta'_\infty$?

A natural generalization of the last question is the balayage problem in the following form where we have replaced δ_∞ by a more general probability measure β. For a given D, does there exist probability measures α and β, $\operatorname{supp}(\alpha) \subset D$, $\operatorname{supp}(\beta) \subset \overline{\mathbb{C}} \backslash \overline{D}$, such that $\alpha' = \beta'$, where the prime denotes balayage onto ∂D? It turns out that stated in this form the balayage problem is strongly connected to the convergence problem for rational interpolation with prescribed poles, to analytic functions, in particular to the so called dual problem; see Section 4, in particular Example 2. The investigation of rational interpolation with prescribed poles, including the dual problem, goes back to work by Walsh [Wa] and Bagby [Ba2], and the present authors have studied it in [Am-Wa1, Am-Wa2], stressing its connection to the balayage problem. In this paper we study the balayage problem stated above, but in a more general formulation (see Subsection 2.2); the main results are stated in Subsection 2.3. These results are applied in Section 4 to the convergence problem for rational interpolation with prescribed poles (see Theorems 3–5; the case $\beta = \delta_\infty$ corresponds to polynomial interpolation).

The general balayage problem is of interest not only because of applications to rational interpolation. It is a natural problem in potential theory and functional analysis, and it is connected to the study of harmonic measures (see [BCGJ] which corresponds to the case when α and β are Dirac measures.

2. RESULTS ON BALAYAGE

2.1. *Balayage.* We define balayage by using the solution to Dirichlet's problem and the Riesz representation theorem. Let Ω be an open *regular* set in $\overline{\mathbb{C}}$; that is each connected component of Ω is regular (for the definition of regular domains, see, for example, [Ra, p. 88]). Then (see [Ra, Corollary 4.1.8]) any continuous function g on $\partial\Omega$ has a unique harmonic continuation u_g inside Ω. Let μ be a finite positive measure with $\operatorname{supp}(\mu) \subset \overline{\Omega}$. We define a linear functional L by

$$g \mapsto Lg = \int u_g \, d\mu$$

on the space $C(\partial\Omega)$ of continuous functions g on $\partial\Omega$. By using the maximum principle for harmonic functions we can easily check that L is bounded. Then, by the Riesz representation theorem [Ru, Chap. 2] there exists a unique measure μ', $\operatorname{supp}(\mu') \subset \partial\Omega$, such that

$$Lg = \int g \, d\mu', \qquad \text{for all} \quad g \in C(\partial\Omega).$$

DEFINITION. μ' is the *balayage* (sweeping out) of μ onto $\partial\Omega$.

From the definition it follows that μ' has the same mass as μ. It also follows that we can get μ' by first sweeping μ onto the boundary of an open subset of Ω, containing $\operatorname{supp}(\mu)$, and then onto $\partial\Omega$. Furthermore, the balayage δ'_a of the Dirac measure δ_a, $a \in D$, is the harmonic measure for a (i.e., evaluated at a) relative to Ω (see for instance [AmWa1, Sect. 2.2]). Finally, δ'_∞ is the equilibrium measure for logarithmic potentials of $\partial\Omega$ if $\infty \in \Omega$ and $\partial\Omega$ is bounded (see [SaTo, Sect. II.4]).

We denote by U_ν the logarithmic potential of a measure ν, $U_\nu = -\int \log |z-t| \, d\nu(t)$. When $\partial\Omega$ and $\operatorname{supp}(\mu)$ are compact subsets of $\mathbb{C}$ the sweeping out process has an interpretation with potentials (see for instance [La, Chap. IV; StTo, Appendix VII]),

$$U_{\mu'}(z) \leqslant U_\mu(z) + c(\mu) \qquad \text{for all} \quad z \in \mathbb{C}, \tag{2}$$

where equality holds for all $z \notin \Omega$ and $c(\mu)$ is a non-negative constant. If Ω is a bounded set in $\mathbb{C}$, $c(\mu) = 0$.

2.2. *The Balayage Problem.* We shall consider a balayage problem for domains. We restrict our discussion to bounded simply connected domains.

Balayage problem for a pair of measures. Let D be a simply connected, bounded domain in $\mathbb{C}$ with closure $\bar{D}$ and boundary ∂D. Let α and β be probability measures in $\bar{\mathbb{C}}$ satisfying $\alpha(D)=1$ and $\beta(\bar{\mathbb{C}}\backslash\bar{D})=1$. For which domains D is it possible to choose α and β so that

$$\alpha' = \beta', \tag{3}$$

where α' and β' are the sweeping out onto ∂D of α and β, respectively?

The condition $\alpha(D)=1$ means that α is concentrated on D. However, $\operatorname{supp}(\alpha)$ may contain points of ∂D. Similarly, $\beta(\bar{\mathbb{C}}\backslash\bar{D})=1$ means that β is concentrated on the complement of $\bar{D}$, but $\operatorname{supp}(\beta)$ may contain points of ∂D. A particularly interesting case is when $\beta=\delta_\infty$, the Dirac measure at infinity. Then β' is the equilibrium measure τ of ∂D (see Subsection 2.1) and (3) takes the form $\alpha'=\tau$.

Remark. There are other, natural versions of the balayage problem for a pair of measures. For instance, assume that we require $\alpha(D)=1$ and $\beta(\bar{\mathbb{C}}\backslash D)=1$, i.e., we allow β or a part of β to be concentrated on ∂D. Then it trivially follows that we can choose α and β so that (3) holds. In fact, we just choose $\beta=\alpha'$. Another related problem is as follows. Let us require that, for some real numbers $a, b\in[0,1]$, $\alpha(D)=a$, $\beta(\bar{\mathbb{C}}\backslash\bar{D})=b$, $\operatorname{supp}(\alpha)\subset\bar{D}$ and $\operatorname{supp}(\beta)\subset\complement D$. Then, again it trivially follows that there are probability measures α and β satisfying (3), if $a+b\leqslant 1$. If $a+b>1$, on the other hand, the problem is equivalent to the original balayage problem for a pair of measures.

2.3. *Results.* Our main result is the following theorem which is proved in Subsection 3.1. The theorem shows that there are probability measures α and β with $\operatorname{supp}(\alpha)\subset D$ and $\operatorname{supp}(\beta)\subset\bar{\mathbb{C}}\backslash\bar{D}$ satisfying (3) if and only if ∂D is an analytic curve.

THEOREM 1. *Let $D\subset\mathbb{C}$ be a bounded simply connected domain. Then the following conditions are equivalent*:

(a) *∂D is an analytic curve.*

(b) *There exit probability measures α and β,* $\operatorname{supp}(\alpha)\subset D$, $\operatorname{supp}(\beta)\subset$ $\bar{\mathbb{C}}\backslash\bar{D}$, *such that $\alpha'=\beta'$, where α' and β' denote the balayage onto ∂D of α and β, respectively.*

(c) *There exists a probability measure α,* $\operatorname{supp}(\alpha)\subset D$, *such that $\alpha'=\delta'_\infty$, where α' and δ'_∞ denote the balayage onto ∂D of α and δ_∞, respectively.*

The next theorem, which is proved in Subsection 3.2, shows that the balayage problem has a negative solution if ∂D is very non-smooth. It is a generalization of a result by Bishop *et al.* on harmonic measure [BCGJ]. They proved the following theorem. Let D be a Jordan domain (i.e., ∂D is a Jordan curve). Let T be the set of all points on ∂D where ∂D has a tangent; see [BCGJ] for the exact definition of a tangent point. (If D is von Koch's snowflake domain, $T = \varnothing$.) Fix two points, z_1 in D and z_2 in the complement of $\bar{D}$, and let ω_j be the harmonic measure for z_j, $j = 1, 2$, relative to D and to the complement of $\bar{D}$, respectively. Denote one dimensional Hausdorff measure by Λ_1. Then $\omega_1 \perp \omega_2$ (i.e., ω_1 and ω_2 are mutually singular) if and only if $\Lambda_1(T) = 0$.

THEOREM 2. *Let D be a Jordan domain such that $\Lambda_1(T) = 0$ where T is the set of tangent points of D. Let α and β be probability measures such that $\alpha(D) = 1$ and $\beta(\bar{\mathbb{C}} \setminus \bar{D}) = 1$. Then $\alpha' \perp \beta'$, where α' and β' denote the balayage onto ∂D of α and β, respectively. In particular $\alpha' \neq \beta'$.*

The following example shows that the balayage problem for a pair of measures may have a positive solution even if ∂D is not an analytic curve.

EXAMPLE 1. Let $\alpha = \frac{1}{2}(\tau_1 + \tau_2)$ where τ_1 is the equilibrium measure of total mass one of the closed interval $[-1, 1]$ and τ_2 the equilibrium measure of total mass one of the closed interval $[-i, i]$ from $-i$ to i on the imaginary axis. Let D be the bounded domain bounded by the level curve $\partial D = \{z \in \mathbb{C} : U_\alpha(z) = U_\alpha(1)\}$. It is possible to show that ∂D has cusps at the points ± 1 and $\pm i$ (D has approximately the form of diamonds in a deck of cards). Then, obviously, the balayage α' of α onto ∂D is the normalized equilibrium measure τ of ∂D. Consequently, if we choose $\beta = \delta_\infty$ we have $\alpha' = \beta'$, and since $\alpha(D) = \beta(\bar{\mathbb{C}} \setminus \bar{D}) = 1$ we have a positive solution of our balayage problem.

3. PROOFS

3.1. *Proof of Theorem* 1. Since the implication (c) $\Rightarrow$ (b) is trivial it is enough to prove (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (c).

Proof of (b) $\Rightarrow$ (a). We first notice that we can get β' by first sweeping β onto ∂D_1, where $D_1 \supset D$ is chosen close to D in the sense that $\operatorname{supp}(\beta) \subset \bar{\mathbb{C}} \setminus \bar{D}_1$, and then sweeping this measure onto ∂D. Because of that we may assume that (b) holds with $\operatorname{supp}(\beta) \subset \mathbb{C} \setminus \bar{D}$. Since $\alpha' = \beta'$ there exists by (2) a constant c such that

$$U_{\beta - \alpha}(z) = U_\beta(z) - U_\alpha(z) = c, \qquad z \in \partial D. \tag{4}$$

We notice that $U_{\beta-\alpha}$ is superharmonic outside supp(α), subharmonic outside supp(β), and harmonic outside supp(α) $\cup$ supp(β). Hence, if we fix a sufficiently small positive number ε, by the maximum principle the sets

$$\Gamma = \{z : U_{\beta-\alpha}(z) = c + \varepsilon\}$$

$$\gamma = \{z : U_{\beta-\alpha}(z) = c - \varepsilon\}$$

have the following properties: $\Gamma \subset \mathbb{C}\backslash\bar{D}$, $\gamma \subset D$, and Γ and γ are closed, non-empty sets without self-intersections. Consequently, Γ and γ are closed Jordan curves in $\mathbb{C}\backslash\bar{D}$ and D, respectively. We fix such an ε and denote by Ω the domain between the curves γ and Γ.

We denote by μ' the balayage of α onto γ and by ν' the balayage of β onto Γ. Since these balayage procedures do not add any mass to the domain Ω, we have the same difference between the values of the potential $U_{\nu'-\mu'}$ on Γ and γ as we had for the potential $U_{\beta-\alpha}$. But for the last potential the difference is $(c+\varepsilon)-(c-\varepsilon)=2\varepsilon$. In addition we note that since $U_{\nu'-\mu'}$ is constant on Γ, this means that ν' is the balayage of μ' onto Γ (cf. for instance [SaTo, Theorem II.4.6]), and this, in turn, means that $U_{\nu'-\mu'}=0$ everywhere on Γ. Consequently, $U_{\nu'-\mu'}=-2\varepsilon$ everywhere on γ.

These arguments show that the pair of probability measures (μ', ν') is the equilibrium distribution for the condenser (γ, Γ) (see [Ba1; SaTo, Chap. VIII; StTo, Appendix VIII], for details on condensers). Then the value $1/(2\varepsilon)$ is called the condenser capacity. The domain Ω is doubly connected. Hence, there exists a one-to-one conformal mapping φ from Ω onto an annulus $\{r<|z|<1\}$, where the positive number r is determined uniquely (see, for example, [Go, Chap. V]).

Now we need the fact that the condenser capacity is unchanged under conformal mappings. This fact follows since the condenser capacity of a doubly connected domain bounded by two curves γ_1 and γ_2 is completely determined by the harmonic function u in the domain with the properties:

(i) u is identically zero on γ_2 (we write $u(\gamma_2)=0$), and identically constant on γ_1 (we denote this value by $u(\gamma_1)$).

(ii) For arbitrary, smooth closed Jordan curves γ_3 in the domain between γ_1 and γ_2, the flux of the vector field grad u across γ_3 is 2π.

These two properties define the harmonic function u uniquely and then the capacity of the condenser (γ_1, γ_2) equals $1/(u(\gamma_2)-u(\gamma_1)) = -1/u(\gamma_1)$.

The argument above in particular shows that the capacities of the condensers (γ, Γ) and $(\{|z|=r\}, \{|z|=1\})$ are equal. We saw above that the capacity of (γ, Γ) is $1/2\varepsilon$. The capacity of $(\{|z|=r\}, \{|z|=1\})$ is determined by the harmonic function $u(z)=\ln|z|$. In fact, it is easy to check that this function satisfies the properties (i) and (ii) above, which means that the capacity of the condenser $(\{|z|=r\}, \{|z|=1\})$ is $1/(\ln 1 - \ln r) = -1/\ln r$. Consequently, we conclude that $2\varepsilon = -\ln r$.

We now consider the harmonic function $\ln|\varphi(z)|$ in Ω, where φ was introduced above. Then $\ln|\varphi(z)| = \ln 1 = 0$ on Γ and $\ln|\varphi(z)| = \ln r = -2\varepsilon$ on γ. This means that the two functions $u_{\nu'-\mu'}(z)$ and $\ln|\varphi(z)|$, harmonic in Ω, coincide on the boundary of Ω and, hence, by uniqueness, in Ω as well. In particular, by balayage and (4), $\ln|\varphi(z)| = u_{\nu'-\mu'}(z) = u_{\beta-\alpha}(z) + c_2 = c_3$ on ∂D, for some constants c_2 and c_3. Hence, $\partial D = \varphi^{-1}(\{|z| = c_3\})$, i.e., ∂D is an analytic curve.

Proof of (a) $\Rightarrow$ (c). Now suppose that (a) is valid, that is ∂D is an analytic curve. Denote by $\psi(z)$ the one-to-one analytic function mapping $\mathbb{C}\setminus\bar{D}$ onto $\{|z|>1\}$. By Schwarz's reflection principle (see for instance [Ra, p. 116]) $\psi(z)$ can be continued to a one-to-one analytic function from a larger domain $A \supset \mathbb{C}\setminus D$ onto $\{|z|>1-\varepsilon\}$ for some fixed $\varepsilon>0$. Introduce, for some fixed r, $1-\varepsilon<r<1$, $\ell = \{z : |\psi(z)| = r\}$, and choose α as the equilibrium measure of ℓ. Denote by B the outer domain of ℓ. Then the Green function $G(z)$ of B with pole at infinity is given by

$$G(z) = c - U_\alpha(z),$$

where c is some constant. On the other hand

$$G(z) = \ln|\psi(z)| - \ln r.$$

From the last equality we get that ∂D is a level curve for $G(z)$ and, consequently, $U_\alpha(z)$ is constant on ∂D. This means that the balayage α' of α onto ∂D is the equilibrium measure of ∂D, i.e., $\alpha' = \delta'_\infty$. If we now choose the points $\{a_{nj}\}_{j=0}^{n}$ on ℓ, asymptotically distributed (as $n\to\infty$) as the measure α, we get (c).

3.2. *Proof of Theorem* 2. For any $z\in D$, let $\omega_z = \omega_{z,\partial D}$ denote the harmonic measure of D evaluated at z. We will first prove that if $a, b \in D$, then

$$c_1\omega_a < \omega_b < c_2\omega_a \tag{5}$$

for some positive constants c_1 and c_2 depending on a and b. In particular, (5) shows that the measures ω_a and ω_b are mutually absolutely continuous.

We first assume that ∂D is an analytic curve. We then let φ be the Riemann mapping function of D onto the unit disk $\{|z|<1\}$ so that $\varphi(a)=0$ for some point $a\in D$. Then φ transforms the harmonic measure ω_a to the normalized Lebesgue measure (i.e., total mass 1) of $\{|z|=1\}$. From this follows, if we denote by ℓ the normalized arc-length measure of ∂D that $c_3\ell < \omega_a < c_4\ell$, for some positive constants c_3 and c_4 depending on a. The last inequality, finally, gives (5).

We now prove that (5) is also true if ∂D is not necessarily analytic. In this case we preliminary enclose the points a, b inside an analytic Jordan curve γ and note that the inequality (5) is true for the harmonic measures $\omega_{a,\gamma}$ and $\omega_{b,\gamma}$ of the domain inside γ. We then observe that the harmonic measures $\omega_{a,\partial D}$ and $\omega_{b,\partial D}$ are the balayage onto ∂D of $\omega_{a,\gamma}$ and $\omega_{b,\gamma}$, respectively. The linearity of the balayage procedure now gives (5) in the general case.

Now let $z_1 \in D$ and $z_2 \in \mathbb{C}\setminus\bar{D}$ be chosen arbitrarily (as in the proof of Theorem 1 we may assume that $\mathrm{supp}(\beta) \subset \mathbb{C}\setminus\bar{D}$). From Theorem 1 in [BCGJ] it follows that $\omega_{z_1} \perp \omega_{z_2}$. Consequently, there are two Borel subsets E_1 and E_2 of ∂D, $E_1 \cap E_2 = \varnothing$, such that $\omega_{z_1}(E_1) = 1$ and $\omega_{z_2}(E_2) = 1$. We have $\omega_{z_1}(\partial D\setminus E_1) = 0$ and from (5) we conclude that the last equality is true if we keep E_1 but change z_1 to any point of D. We combine this with the following fact. If α is any measure in D and α' its balayage onto ∂D, we have (see for instance [SaTo, Sect. II.4])

$$\alpha' = \int_D \omega_z \, d\alpha(z).$$

The conclusion is that $\alpha'(\partial D\setminus E_1) = 0$ and, consequently, $\alpha'(E_1) = 1$. Analogously we get that if β is any measure in $\bar{\mathbb{C}}\setminus\bar{D}$, then $\beta'(E_2) = 1$. Summarizing, we have $\alpha'(E_1) = \beta'(E_2) = 1$ and $E_1 \cap E_2 = \varnothing$. This proves that $\alpha' \perp \beta'$ and, in particular, $\alpha' \neq \beta'$.

4. APPLICATIONS TO RATIONAL INTERPOLATION

Our assumptions in this section are as follows. Let $D \in \bar{\mathbb{C}}$ be a regular domain with boundary ∂D. Let $A_n = \{a_{nj}\}_{j=0}^{n} \subset D$, the interpolation points, and $B_n = \{b_{nj}\}_{j=1}^{n} \subset \bar{\mathbb{C}}$, the poles, be two sets of points such that $A_n \cap B_n = \varnothing$ and $\bigcup_{n \geqslant 1} B_n$ has no limit point in D. Let f be an analytic function in D. Then there exists [Wa, Sect. 8.1] a unique rational function $r_n = P_n/Q_n$ of degree n (i.e., P_n and Q_n are polynomials of degree at most n) with poles at B_n interpolating to f at A_n, i.e., r_n has prescribed poles at the points of B_n and $r_n = f$ at the points of A_n, counting multiplicities. We introduce the normalized point counting measure of B_n:

$$\beta_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{b_{nj}}. \tag{6}$$

Let α and β be weak star limit points of the sets $\{\alpha_n\}_1^\infty$ and $\{\beta_n\}_1^\infty$, respectively, where α_n and β_n are the normalized point counting measures given by (1) and (6). We observe that if $b_{nj} = \infty$ for all j and n, r_n becomes the interpolating polynomial P_n discussed in Section 1. In this case $\beta_n = \beta = \delta_\infty$.

In [Am-Wa1] we proved the following convergence theorem for the interpolating rational function r_n.

THEOREM 3 [Am-Wa1, Theorems 1 and 2]. *Assume that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D. Then $r_n \to f$ in D, as $n \to \infty$, for every analytic function f in D, if and only if $\alpha' = \beta'$ for any weak star limit points α and β, of $\{\alpha_n\}_1^\infty$ and $\{\beta_n\}_1^\infty$, respectively, and in that case the convergence is locally uniform with geometric degree of convergence.*

EXAMPLE 2. We claim that by combining Theorems 3 and 1 we get the following result. Let D be a bounded, simply connected domain. Then ∂D is an analytic curve if and only if there exist $A_n \subset D$ and $B_n \subset \overline{\mathbb{C}}$ such that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D and $\bigcup_{n \geqslant 1} B_n$ no limit point on $\bar{D}$, and such that the rational function r_n with poles B_n interpolating to f at A_n converges to f, as $n \to \infty$, for every analytic function f in D. In fact, the if part follows immediately from Theorems 3 and 1. In the proof of the only if part we can, by Theorem 1, assume that there exists a probability measure α, $\operatorname{supp}(\alpha) \subset D$, such that $\alpha' = \delta'_\infty$. We choose $b_{nj} = \infty$ for all j and n, which gives $\beta_n = \beta = \delta_\infty$. By a standard argument we can discretize α to find $A_n \subset D$, so that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D and $\alpha_n \to \alpha$ in the weak star sense. Since $\alpha' = \delta'_\infty$ an application of Theorem 3 shows that in this case $r_n \to f$ for all analytic functions f in D. Observe that with this choice of B_n, r_n has all its poles at infinity, i.e., r_n is a polynomial. Consequently, if ∂D is an analytic curve, there exist $A_n \subset D$, $n \geqslant 1$, so that $\bigcup_{n \geqslant 1} A_n$ has no limit point on ∂D and the corresponding polynomials P_n converge to f for all analytic functions f in D.

If we consider sets $\bigcup_{n \geqslant 1} A_n \subset D$ and $\bigcup B_n \subset \overline{\mathbb{C}} \backslash \bar{D}$ having no limit point on ∂D, the condition $\alpha' = \beta'$ in Theorem 3 means that there is a certain duality between interpolation points and poles. For the convergence $r_n \to f$ it does not matter if we consider rational functions with poles in B_n interpolating at A_n and analytic functions f in D, or rational functions with poles in A_n interpolating at B_n and analytic functions f in $\overline{\mathbb{C}} \backslash \bar{D}$ (we then have to work with the same number of points in A_n and B_n and adapt the problem to this); see [Wa, Ba2] for details on the duality.

We finish Example 2 by remarking that by Theorem 3 we get convergence $r_n \to f$ in D, for all analytic functions f in D, for any choice of $A_n \subset D$ such that $\bigcup_{n \geqslant 1} A_n$ has no limit point in D, if we take b_{nj} on ∂D so that $\beta = \alpha'$.

If $\bigcup_{n \geqslant 1} A_n$ has limit points on ∂D we get a weaker result than Theorem 3 as shown by the next two theorems.

THEOREM 4 [Am-Wa2, Theorem 1]. *Let ∂D be bounded and assume that $\alpha(D) > 0$ for any weak star limit point α of $\{\alpha_n\}_1^\infty$, and that*

$$\lim_{n\to\infty} \left[\sup_{z\in\partial D} \left(U_{\alpha_n'}(z) - U_{\beta_n'}(z) \right) \right] = 0. \tag{7}$$

Then $r_n \to f$ in D, as $n \to \infty$, for any bounded, analytic function f in D, locally uniformly with geometric degree of convergence.

THEOREM 5. *Let $D \subset \mathbb{C}$, $D \neq \mathbb{C}$ be a simply connected domain. Assume that $\bigcup_{n\geqslant 1} A_n$ has at least one limit point on ∂D. Then, for an arbitrary point $z_0 \in D \setminus \bigcup_{n\geqslant 1} A_n$, there exists an analytic function f in D such that, for the corresponding rational interpolants r_n, we have*

$$\limsup_{n\to\infty} |f(z_0) - r_n(z_0)| = \infty.$$

Proof of Theorem 5. We prove the theorem in the more general case when B_n is any point set satisfying $A_n \cap B_n = \emptyset$. When D is a disk Theorem 5 is Theorem 2 in [Am-Wa2]. We shall adapt the proof for a general domain D to the case of a disk by a conformal mapping. Suppose that $z_0 = 0$. We shall construct f of the form $f(z) = zg(z)$ where g is analytic in D.

Step 1. As in [Am-Wa2], Formulas (17) and (18), we conclude that $f(0) - r_n(0) = \sum_{j=1}^n c_{nj} g(a_{nj})$, where $c_{nj} \neq 0$ depend only on $\{A_n, B_n\}$ and not on g.

Step 2. We want to construct g so that $f(0) - r_n(0) \to \infty$, as $n \to \infty$. In [Am-Wa2] this was done when D is a disk. We use that result by making a conformal mapping ϕ of D onto the unit disk D^* and solving the analogous problem in D^* with interpolation points $a_{nj}^* = \phi(a_{nj})$ and poles $b_{nj}^* = \phi(b_{nj})$ (see [Am-Wa2, Sect. 4]). This gives an analytic function g^* in D^* such that $g = g^* \circ \phi$ solves our problem.

EXAMPLE 3. Let D be a rectangle. From Theorem 5 and Example 2 we conclude that there does not exist $A_n \subset D$ such that the polynomial P_n interpolating to f at A_n converges to f in D for all analytic functions f in D.

In our last example we shall need the following lemma which is easily verified.

LEMMA. *Let τ_1 be the probability equilibrium measure of $[-1, 1]$ and $I = [-1+\ell, 1-\ell]$, where ℓ is fixed, $0 < \ell < 1/2$. Let μ_m be the normalized point counting measure of the set of points of I consisting of zeros of the mth Chebyshev polynomial on $[-1, 1]$. Let μ be the restriction of τ_1 to I,* *normalized so that μ is a probability measure. Then $\mu_m \to \mu$ in the weak star sense and $U_{\mu_m} \to U_\mu$, uniformly on ∂D, as $m \to \infty$.*

EXAMPLE 4. Let D be the domain in Example 1. As in the previous example, by Theorem 5 and Example 2 we observe that there does not exist $A_n \subset D$ such that the polynomial P_n interpolating to f at A_n converges to f for all analytic functions f in D. However, we claim that we can use Theorem 4 to conclude that it is possible to choose $A_n \subset D$ so that the interpolating polynomial P_n converges to f for any *bounded* analytic function f in D.

In fact, in the notation of Example 1, $\alpha = \frac{1}{2}(\tau_1 + \tau_2)$ and $\alpha' = \tau$. We choose $b_{nj} \equiv \infty$ which means that $\beta_n = \beta = \delta_\infty$ and $\beta'_n = \beta' = \tau$. In order to find α_n we shall discretize α in such a way that (7) holds. Due to balayage and the fact that $\beta'_n = \tau = \alpha'$, (7) can be written

$$\lim_{n \to \infty} \left[\sup_{z \in \partial D} \left(U_{\alpha_n}(z) - U_\alpha(z) \right) \right] = 0. \tag{8}$$

To find α_n satisfying (8) we first discretize τ_1 by using the lemma with numbers $\ell = \ell_k$, $k = 1, 2, ...,$ tending to zero, and $I = I_k$. We then discretize τ_2 in an analogous way. For a fixed k, starting with $k = 1$, we now choose α_n for a finite sequence of values of n, by using the discretization of the restriction to I_k of τ_1 and the corresponding discretization of τ_2. We then go from k to $k+1$ and repeat the process. Because of the lemma we can obtain α_n, $n = 1, 2, ...,$ satisfying (8). Furthermore, $\alpha_n \to \alpha$ in the weak star sense and $\alpha(D) > 0$, i.e., our claim follows from Theorem 4.

REFERENCES

[AmWa1] A. Ambroladze and H. Wallin, Rational interpolants with pre-assigned poles, theory and practice, *Complex Variables* **34** (1997), 399–413.

[AmWa2] A. Ambroladze and H. Wallin, Rational interpolants with pre-assigned poles, theoretical aspects, *Studia Math.* **132** (1999), 1–14.

[Ba1] T. Bagby, The modulus of a plane condenser, *J. Math. Mech.* **17** (1967), 315–329.

[Ba2] T. Bagby, Rational interpolation with restricted poles, *J. Approx. Theory* **7** (1973), 1–7.

[BCGJ] C. J. Bishop, L. Carleson, J. B. Garnett, and P. W. Jones, Harmonic measures supported on curves, *Pacific J. Math.* **138** (1989), 233–236.

[Go] G. M. Goluzin, "Geometric Theory of Functions of a Complex Variable," 2nd ed., Nauka, Moscow, 1966; English translation, Amer. Math. Soc., Providence, 1969.

[La] N. S. Landkof, "Foundations of Modern Potential Theory," Springer-Verlag, New York, 1972.

[Ra] T. Ransford, "Potential Theory in the Complex Plane," Cambridge Univ. Press, Cambridge, 1995.

[SaTo] E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, Berlin, 1997.

[StTo] H. Stahl and V. Totik, "General Orthogonal Polynomials," Cambridge Univ. Press, Cambridge, UK, 1992.

[Wa] J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 20, Amer. Math. Soc., Providence, RI, 1965.